Comparing different versions of tiling cohomology

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Abstract

We establish direct isomorphisms between different versions of tiling cohomology. The first version is the direct limit of the cohomologies of the approximants in the Anderson-Putnam-Gähler system, the second is the recently introduced PV-cohomology of Savinien and Bellissard and the third is pattern equivariant cohomology. For the last two versions one can define weak cohomology groups. We show that the isomorphisms extend to the weak versions. This leads to an alternative formulation of the pattern equivariant mixed quotient group which describes deformations of the tiling modulo topological conjugacy.

1 Introduction.

There exist various isomorphic pictures of what one now calls tiling cohomology. Originally it was defined as the groupoid cohomology of the tiling groupoid [8] or as the Čech cohomology of the tiling space (the hull) [1]. If the tiling space can be seen as an inverse limit of finite simplicial (or cellular) complexes. the cohomology of the tiling space is a direct limit of simplicial (or cellular) cohomology groups [1], [5, 16]. An intuitively simpler version was later proposed using the concept of pattern equivariant exterior forms [9, 10] and then modified using pattern equivariant simplicial (or cellular) cochains [17]. Finally, a recent version making full use of the simplicial structure defined by the tiling goes under the name of PV-cohomology [18]. Each version has his own advantages and disadvantages. Computationally the groupoid cohomology approach was most successful for cut & and project patterns [4] and the direct limit approach for substitution tilings [1]. The newer versions led conceptionally to new insight, especially the pattern equivariant de Rham cohomology. The latter comes about as it naturally offers different choices of cohomology groups based on the distinction between algebraic modules and their topological closure. The original tiling cohomology is isomorphic to strongly pattern equivariant cohomology, the algebraic version. The topologically closed version is the so-called weakly pattern equivariant cohomology, and this one is isomorphic to the tangential cohomology of the hull (seen as a lamination) [10]. It has been shown that the theory of deformations of tilings is related to the mixed group of strongly pattern equivariant cochains modulo weakly pattern equivariant coboundaries [11]. It is therefore of quite some interest to determine the latter. However, apart from substitution tilings [3], we have at present no general method to offer to compute this mixed group.

The present article sheds new light on the different versions of pattern equivariant cohomology in that it provides analogues of the weak and the mixed version in the framework of PV-cohomology. To obtain these we provide explicit descriptions of the isomorphisms between the various cohomology groups. We establish in particular an explicit formula for the isomorphism between pattern equivariant de Rham cohomology and PV-cohomology and show that it carries the right continuity properties for it to extend from the algebraic to the topological closed versions of the cohomology groups.

2 Preliminaries.

In this section we provide some known background material on tilings. In particular we recall the definitions of the following versions of tiling cohomology:

1. as direct limit of the simplicial cohomologies of the Anderson-Putnam-Gähler complexes [1, 5, 16] (we refer to that simply as DL-cohomology),
2. as PV cohomology [18],
3. as PE de Rham cohomology [9, 10],
4. as PE simplicial cohomology [17].

We also recall or introduce weak versions of PV and PE cohomology. PE stands for *pattern equivariant*, s-PE for *strongly* and w-PE for *weakly* pattern equivariant.

2.1 Tilings and tilings spaces

A tiling of the euclidean space $\mathbb{R}^d$ is a covering of the space by closed topological disks (its tiles) which overlap at most on their boundaries. We consider here simplicial tilings of (translational) finite local complexity. This means that the tiles are simplices of $\mathbb{R}^d$ which touch face to face and that there are only finitely many simplices up to translation; we present more details in Section 2.1.1. In the context of tilings of finite local complexity the restriction to simplicial tilings is topologically not a restriction, since there are ways to associate to any tiling of finite local complexity a simplicial tiling which carries the same topological information.

The tiling space or hull of a tiling $\mathcal{P}$ is

$$\Omega_{\mathcal{P}} := \overline{\{\mathrm{T}^x(\mathcal{P}) | x \in \mathbb{R}^d\}}^D,$$

the completion of the translation orbit of $\mathcal{P}$ w.r.t the metric given by

$$D(\mathcal{P}_1, \mathcal{P}_2) = \inf_{r>0} \{ \frac{1}{r+1} | \exists x, y \in B_{\frac{1}{r}}(0) : B_r[\mathrm{T}^x(\mathcal{P}_1)] = B_r[\mathrm{T}^y(\mathcal{P}_2)] \}$$

where $\mathcal{P}_1$ and $\mathcal{P}_2$ are tilings of $\mathbb{R}^d$. Here T^x denotes the translation action $\mathrm{T}^x(Q) = Q - x$ where Q is a point or a geometric object and we use the notation $B_r[\mathcal{P}] = B_r[\mathcal{P}']$ to indicate that the tiling $\mathcal{P}$ and $\mathcal{P}'$ coincide[1] on the r-ball around 0.

Tiling spaces can be obtained as inverse limits of simpler spaces. Such a construction was given for substitution tilings by [1] and then later generalized to all tilings of finite local complexity in [5, 16] and a somewhat similar construction given in [2]. We present the details of the construction using k-neighborhoods.

Let $\mathcal{P}$ be a tiling of $\mathbb{R}^d$ and t be the tile (or the patch) of $\mathcal{P}$. The first corona of t is the set of its nearest neighbors, the second corona of t is the set of the second nearest neighbors, and so on out to the k^{th} corona. The k-neighborhood of a tile is the collection of its jth coronae for $j \leq k$. Consider two tiles t_1, t_2 in $\mathcal{P}$, we say that t_1 is equivalent to t_2, if t_1 is translationally congruent to t_2, the equivalence class is called prototile. The condition of finite local complexity implies that there are finitely many prototiles. A k-collared tile is a tile labelled with its k-neighborhood. The labelling of the tile does not change the set covered by the tile, which we call its support, but it is only a decoration of it. A k-collared prototile is the translational congruence class of the k-collared tile. Again, finite local complexity implies that the number of k-collared prototiles is finite. Let f_1 and f_2 to be faces of two k-collared prototiles $\tilde{t}_1$ and $\tilde{t}_2$. We say f_1 is equivalent to f_2 if $\tilde{t}_1$ has a representative t_1 and $\tilde{t}_2$ has a representative t_2 in $\mathcal{P}$ such that the edges corresponding to f_1 and f_2 coincide. The disjoint union of the supports of the k-collared prototiles quotiened out by this equivalence relation on the faces is the set Γ_k. Equipped with the quotient topology we consider it as a topological space. It inherits the structure of a complex from $\mathcal{P}$ and is therefore called an Anderson-Putnam-Gähler complex.

Let $\rho_{k+1\,k} : \Gamma_{k+1} \longrightarrow \Gamma_k$ the map which associates to a point of a $k+1$-collared prototile the same point in the k-collared prototile obtained

[1] e.g. their sets of boundary points of their tiles coincides on the r-ball around 0

by forgetting about the $(k+1)^{th}$ corona in its label. The inverse system $(\Gamma_k, \rho_{k\,k-1})_{k\in\mathbb{N}}$ is called the Anderson-Putnam-Gähler system. Its importance lies in the fundamental result that its inverse limit is homeomorphic to the tiling space

$$\Omega_{\mathcal{P}} \cong \varprojlim_{k} \Gamma_k.$$

A point of Γ_k tells us how to place a patch corresponding to a k-collared tile around the origin of $\mathbb{R}^d$. By definition of the inverse limit the surjections $\rho_{k\,k-1}$ induce a continuous surjection

$$\rho_k : \Omega_{\mathcal{P}} \longrightarrow \Gamma_k \tag{2.1}$$

which can be described as follows. Consider the tile of $\omega \in \Omega_{\mathcal{P}}$ together with its k-neighborhood which lies on the origin $0_{\mathbb{R}^d}$. Taking congruence classes w.r.t. translations, the point in the tile above corresponds to a point in a k-collared prototile. $\rho_k(\omega)$ is this point viewed as a point in Γ_k. By the definition of the equivalence for Γ_k, this is well defined also in the cases in which a boundary point of a tile lies on the origin. Related to the above is the surjection $\tilde{\rho}_k : \mathbb{R}^d \longrightarrow \Gamma_k$

$$\tilde{\rho}_k(x) = \rho_k(\mathrm{T}^x(\mathcal{P})). \tag{2.2}$$

2.1.1 Simplicial tilings and Δ-complexes

For their definition of PV-cohomology Savinien and Bellissard consider tilings whose tiles are finite Δ-complexes which are compatible in the sense that the intersection of two tiles are sub-Δ-complexes. Intuitively the reader should simply have in mind a tiling whose tiles are simplices which meet face to face and such that simplices can be pattern equivariantly oriented. We call such tilings simplicial tilings.

In more technical tems, a Δ-complex structure (see [6]) on a topological space X is a collection of continuous maps $\sigma : \Delta^n \to X$, where

$$\Delta^n = \{(x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} x_i = 1, x_i \geq 0\}$$

is the standard n-simplex in euclidean space. It is supposed that these maps are injective on the interior of Δ^n. They are referred to as the characteristic maps and their images are the so-called cells (of dimension n) of X. It is also supposed that each point of X lies in the interior of exactly one cell. Δ^n has $n+1$ faces of dimension $n-1$, we write $\partial_i\Delta^n$ for the face described by coordinates $(x_1, x_2, \cdots, x_{n+1})$ for which $x_i = 0$ and call it the ith face. We write $\partial_i\sigma$ for the restriction of σ to $\partial_i\Delta^n$. It is also a characteristic map, but for an $n-1$ cell. When cells are next to each other, touching along lower dimensional faces, or when characteristic functions are not injective on the boundary but identify different faces, then the corresponding characteristic maps of the lower dimensional faces are identified. The vertices of Δ^n are ordered, the ith vertex being the one having coordinate $x_i = 1$, and this order passes to the faces. It is assumed that the identification of characteristic maps preserves the ordering of the vertices. They are further compatibility conditions among the characteristic maps for which we refer to [6, 18]. We will use the phrasing "the simplex σ of X" to refer to the characteristic map or to the cell it defines in X.

A simplicial complex is special kind of Δ-complex. In [19] this notion is used for Δ-complexes discribing subsets of euclidean spaces whose cells are actually simplices, i.e. convex hulls of $n+1$ vertices (in general position) if n is their dimension.

Lemma 2.1. *Any tiling of finite local complexity is mutually locally derivable with a tilings whose tiles are the top-dimensional simplices of a simplicial complex. Moreover, the simplicial structure is pattern equivariant in the following sense. If t_1 and t_2 are tiles which differ by a translation, i.e. $t_2 = \mathrm{T}^x(t_1)$ for some x, then the characteristic functions σ_1 and σ_2 of t_1 and t_2, respectively, differ by the same translation, $\sigma_2 = \mathrm{T}^x \circ \sigma_1$.*

Proof. It is well known that any tiling of finite local complexity is mutually locally derivable with a tiling whose tiles are polyhedra and match face to face. One may then subdivide the polyhedra into simplices in a mutually locally derivable way to obtain a tiling whose tiles are simplices and match face to face. The issue is now to order the vertices of the simplices in a pattern equivariant way. This can be done as follows: By the finite local complexity the exists a unit vector ν in $\mathbb{R}^d$ (the ambient space) such that none of the edges of the tiles is perpendicular to ν. Then the order on the vertices defined by $v_1 \leq v_2$ provided $\langle \nu, v_1 \rangle \leq \langle \nu, v_2 \rangle$ is total and translation invariant in the sense that the pair $(v_1 - x, v_2 - x)$ (if it occurs in the tiling) has the same order than (v_1, v_2). We may therefore define the characteristic maps to be translation invariant in the same sense. □

We call a tiling with this property a simplicial tiling. We denote by $\mathcal{P}^n$ the n-skeleton of the simplicial complex it defines, that is, $\mathcal{P}^n$ is the union of all faces of dimension smaller or equal to n. In particular $\mathcal{P}^d = \mathbb{R}^d$.

If X carries in addition the structure of a differentiable manifold and we want to integrate n-forms over certain n-dimensional submanifolds it is useful to use charts defined by the cells. For that purpose we shall not only require the characteristic maps to be diffeomorphisms, but even that they extend to diffeomorphisms from an open neighborhood of Δ^n to an open neighborhood of the corresponding cell; in the terminologie of [19] this means that the simplicial complex defines a smooth triangulation of X. This is clearly possible for the simplicial complex defined by a tiling in $\mathbb{R}^d$.

The tiling defines a Δ-complex for the topological spaces $X = \Gamma_k$. This is obvious from the construction of Γ_k. In fact, if σ is a characteristic map of a k-collared tile then $\tilde{\sigma} := \tilde{\rho}_k \circ \sigma$ is a characteristic map for the corresponding k-collared prototile in Γ_k. By Lemma 2.1 this is independent of the choice of representative for the prototile. Note that $\tilde{\sigma}$ is not necessarily injective. We denote by $\mathcal{S}_k^n$ the set of characteristic maps of the n-simplices on Γ_k.

An n-chain on X is a formal linear combination of oriented simplices from X, and we denote the set of n-chains by $C_n(X)$. It is thus the free $\mathbb{Z}$-module generated by elements $\langle\sigma\rangle$. An n-cochain with coefficients in an abelian group A is an element of the $\mathbb{Z}$-module $C^n(X, A) = \mathrm{Hom}_{\mathbb{Z}}(C_n(X), A)$. Let $a \in A$. We denote by a_σ the morphism defined by

$$a_\sigma(\langle\tau\rangle) = \begin{cases} a & \text{if } \sigma = \tau \\ 0 & \text{else} \end{cases}$$

.

2.1.2 The Δ-transversal of a tiling

The canonical transversal or discrete hull of a tiling is defined with the help of punctures for the tiles of the tiling [7]. In [18] punctures for the faces of tiles are introduced as well so that one gets a complex defined by transversals of different dimension.

A puncture of a face of a tile is a chosen point in the face, for instance its barycenter. The pair given by the face and its puncture is called a punctured face. All punctures are supposed to be chosen in such way that two faces of tiles in the tiling which agree up to translation have their punctures at the same relative position. Then the puncturing passes to the equivalence classes so that the simplices of Γ_k are also punctured. We denote by $\mathcal{P}^{n,punc}$ the set of punctures of all n-faces of $\mathcal{P}$.

Definition 2.2. *[18] The n-dimensional Δ-transversal is*

$$\Xi^n = \{\mathcal{P}' \in \Omega_{\mathcal{P}} | 0 \in \mathcal{P}'^{n,punc}\}.$$

The disjoint union $\Xi = \dot{\bigcup}_{n=0}^{d} \Xi^n$ is called the Δ-transversal of the tiling $\mathcal{P}$.

In [18] the notation Ξ_Δ for the Δ-transversal was used instead which distinguishes it from the usual notation for the canonical transversal, but we see no danger of confusion here. The sets Ξ^n are totally disconnected and compact subsets of $\Omega_{\mathcal{P}}$. Their topology is generated by the following clopen subsets, referred to as acceptance zones for faces of Γ_k.

Definition 2.3. *The acceptance zone of an n-simplex σ of Γ_k is the subset $\Xi(\sigma)$ of Ξ^n given by*

$$\Xi(\sigma) = \rho_k^{-1}(p_\sigma)$$

where p_σ is the puncture of σ. The surjections ρ_k are defined in (2.1).

Remark 2.4. *Recall that a d-simplex σ in Γ_k corresponds to (the congruence class of) a tile t_σ in $\mathcal{P}$ labelled with its k-neighborhood. In a similar fashion a n-simplex σ in Γ_k corresponds to an n-dimensional face of a tile in $\mathcal{P}$ together with its k-neighborhood. Then $\Xi(\sigma)$ can be seen as the subset of tilings which contain a patch corresponding to a k-collared face defined by σ whose puncture lies on $0 \in \mathbb{R}^d$.*

We denote the restriction of D to Ξ by D_0. The family of clopen balls $\{\mathcal{B}_{D_0}(\mathcal{P}, \varepsilon)\}_{\mathcal{P}\in\Xi}$, of radius ε around $\mathcal{P}$, is a base of the topology of the space (Ξ, D_0). We denote $\mathcal{B}^n_{D_0}(\mathcal{P}, \varepsilon) = \mathcal{B}_{D_0}(\mathcal{P}, \varepsilon) \cap \Xi^n$.

Lemma 2.5. *The family of acceptance zones $\{\Xi(\sigma)\}_{\sigma\in\Gamma_k, 0\leq k\leq d}$ is a base for the topology of the metric space (Ξ, D_0).*

Proof. Let σ be an n-simplex of Γ_k and $\mathcal{T}$ a tiling in $\Xi^n(\sigma)$. Recall that $D_0(\mathcal{T}, \mathcal{T}') \leq \frac{1}{R+1}$ means that $\mathcal{T}$ and $\mathcal{T}'$ agree on the ball $B_R(0)$ of $\mathbb{R}^d$ of radius R around the origin. Choose $R > 0$ large enough such that $B_R(0)$ covers the k-neighborhood of the face of $\mathcal{T}$ at the origin. Then $D_0(\mathcal{T}, \mathcal{T}') \leq \frac{1}{R+1}$ implies that $\mathcal{T}'$ contains as well that face with its k-neighborhood and hence belongs to the acceptance zone $\Xi(\sigma)$. We thus have proved $\exists R > 0$: $\mathcal{T} \in \mathcal{B}^n_{D_0}(\mathcal{T}, \frac{1}{R+1}) \subset \Xi(\sigma)$.
Conversely, let $\mathcal{T}$ be a tiling of Ξ^n and $R > 0$. Choose k large enough so that the k-neighborhood of the face f at the origin of $\mathcal{T}$ covers $B_R(0)$. Let $\sigma_f \in \Gamma_k$ be the n-simplex corresponding to this k-collared face. Then $\mathcal{T}' \in \Xi(\sigma_f)$ implies that $\mathcal{T}$ and $\mathcal{T}'$ agree on $B_R(0)$ and hence $D_0(\mathcal{T}, \mathcal{T}') \leq \frac{1}{R+1}$. We thus have proved that there exists an acceptance zone $\Xi(\sigma_f)$ such that $\mathcal{T} \in \Xi(\sigma_f) \subset \mathcal{B}^n_{D_0}(\mathcal{T}, \frac{1}{R+1})$. $\square$

We denote by $\mathcal{C}_{lc}(\Xi^n, A)$ the $\mathbb{Z}$-module of A-valued locally constant functions over the n-dimensional Δ-transversal. It constitutes the degree n part of the complex defining PV cohomology which we will consider below.

It follows from Lemma 2.5 that any element of $\mathcal{C}_{lc}(\Xi^n, A)$ is a finite sum of elements $a_{\Xi(\sigma)}$ where $a \in A$ and

$$a_{\Xi(\sigma)}(\xi) = \begin{cases} a & \text{if } \xi \in \Xi(\sigma) \\ 0 & \text{else} \end{cases} .$$

2.2 Cohomology as a direct limit (DL-cohomology)

As mentioned in the introduction, one way to define the cohomology of a tiling $\mathcal{P}$ is as Čech cohomology $\check{H}(\Omega_\mathcal{P}, A)$ of $\Omega_\mathcal{P}$. Since $\Omega_\mathcal{P}$ is an inverse limit of spaces $(\Gamma_k)_{k\in\mathbb{N}}$ its Čech cohomology can be expressed as a direct limit of groups

$$\check{H}(\Omega_\mathcal{P}, A) = \check{H}(\varprojlim_l \Gamma_l, A) = \varinjlim_l \check{H}(\Gamma_l, A).$$

The main advantage is that $\check{H}(\Gamma_k, A)$, the Čech cohomology of Γ_k, is simple to compute. As Γ_k is a finite Δ-complex its Čech cohomology is isomorphic to its simplicial cohomology. Furthermore, the maps $\rho_{k\,k-1}$ preserve the Δ-complex structure and so the maps they induce on simplicial cohomology can easily be handled. In its original version this was the method to calculate the tiling cohomology for substitution tilings in [1] (although they used cellular cohomology).

2.3 Savinien and Bellissard's approach to tiling cohomology

2.3.1 PV-cohomology.

Recall that $\mathcal{S}_0^n$ denotes the (finite) set of characteristic maps of the n-simplices in Γ_0. Let $\tilde{\sigma} \in S_0^n$ and choose a characteristic map σ of an n-simplex of $\mathcal{P}$ such that $\tilde{\sigma} = \tilde{\rho}_0 \circ \sigma$. We define the vector

$$x_{\tilde{\sigma},i} = p_{\partial_i \sigma} - p_\sigma \in \mathbb{R}^d$$

which is the difference vector pointing from the puncture of the cell defined by σ to the puncture of its ith face. This is independent of the choice of σ by Lemma 2.1. We denote by $\mathrm{T}_{\tilde{\sigma},i} : \Xi(\tilde{\sigma}) \to \Xi(\partial_i \tilde{\sigma})$ the restriction of $\mathrm{T}^{x_{\tilde{\sigma},i}}$ to $\Xi(\tilde{\sigma})$, $\mathrm{T}^{x_{\tilde{\sigma},i}}(\xi) = \xi + p_\sigma - p_{\partial_i \sigma}$. The following definition is essentially that of [18].

Definition 2.6. *Let $\tilde{\sigma} \in S_0^n$. Define the linear map $\theta_{\tilde{\sigma},i} : C_{lc}(\Xi^{n-1}, A) \to C_{lc}(\Xi^n, A)$ by*

$$\theta_{\tilde{\sigma},i}(f) = \imath \circ \mathrm{T}^*_{\tilde{\sigma},i}(f \left|_{\Xi(\partial_i \tilde{\sigma})}\right.)$$

where $\mathrm{T}^*_{\tilde{\sigma},i} : C_{lc}(\Xi(\partial_i \tilde{\sigma}), A) \to C_{lc}(\Xi(\tilde{\sigma}), A)$ is the pull back of $\mathrm{T}_{\tilde{\sigma},i}$ and $\imath : C_{lc}(\Xi(\tilde{\sigma}), A) \hookrightarrow C_{lc}(\Xi^n, A)$ the inclusion. The strong PV-cohomology of $\mathcal{P}$ is the cohomology of the complex $(C_{lc}(\Xi^*, A), d_{PV})$ where $d_{PV} : C_{lc}(\Xi^n, A) \longrightarrow C_{lc}(\Xi^{n+1}, A)$ is given by

$$d_{PV} = \sum_{\tilde{\sigma} \in \mathcal{S}_0^{n+1}} \sum_{i=0}^{n+1} (-1)^i \theta_{\tilde{\sigma},i}.$$

We denote the strong PV-cohomology with coefficients in A by $H^*_{PV}(\Gamma_0, \mathcal{C}_{lc}(\Xi, A))$.

$H^*_{PV}(\Gamma_0, \mathcal{C}_{lc}(\Xi, \mathbb{Z}))$ is isomorphic to the Čech cohomology of the hull [18].

2.3.2 Relation with pattern groupoid cohomology

PV-cohomology can effectively seen as the continuous cocycle cohomology (after [15]) of the pattern groupoid $\mathcal{GP}$, which is the variant of the discrete tiling groupoid defined in [4]. More precisely, the PV-complex is a reduction of the complex of continuous cocycle cohomology which has the advantage of being trivial in degrees larger than the dimension of the tiling. This is similar to using a finite resolution of length d instead of the standard resolution for the cohomology of the group $\mathbb{Z}^d$. Let us explain this.

By definition, $\mathcal{GP}$ is the reduction to $\Xi^{(0)}$ of the transformation groupoid $\Omega_{\mathcal{P}} \rtimes \mathbb{R}^d$. Its elements are pairs $(\omega, v) \in \Omega_{\mathcal{P}} \times \mathbb{R}^d$ which satisfy $\omega, \mathrm{T}^v(\omega) \in \Xi^{(0)}$. The continuous cocycle cohomology of $\mathcal{GP}$ is the cohomology of the complex of continuous functions

$$C(\mathcal{GP}^{(n)}, \mathbb{Z}) \xrightarrow{\delta} C(\mathcal{GP}^{(n+1)}, \mathbb{Z}) \cdots$$

where $\mathcal{GP}^{(0)} = \Xi^{0)}$ and $\mathcal{GP}^{(n)}$ is the subset of elements $(\omega, v_1, \cdots, v_n) \in \Omega_{\mathcal{P}} \times (\mathbb{R}^d)^n$ which satisfy $\omega \in \Xi^{(0)}$ and $\omega - \sum_{i=1}^{j} v_i \in \Xi^{(0)}$ for all j. Its topology is the subspace topology of the product topology. The differential is given by $\delta f(\omega, v) = f(\mathrm{T}^v(\omega)) - f(\omega)$ in degree 0, and in degree $n > 0$ by

$$\begin{aligned} \delta f(\omega, v_1, \cdots, v_{n+1}) &= f(\mathrm{T}^{v_1}(\omega), v_2, \cdots, v_{n+1}) + (-1)^{n+1} f(\omega, v_1, \cdots, v_n) \\ &\quad + \sum_{i=1}^{n} (-1)^i f(\omega, v_1, \cdots, v_i + v_{i+1}, \cdots, v_{n+1}) \end{aligned}$$

We define the map $\alpha : \Xi^{(n)} \to \mathcal{GP}^{(n)}$ by

$$\alpha(\omega) = (\omega, \sigma(x_1) - \sigma(x_0), \cdots, \sigma(x_n) - \sigma(x_{n-1}))$$

where $\sigma(x_i)$ is the ith vertex of the n-simplex of ω whose puncture is on 0. Clearly α is injective. It is continuous as the $\sigma(x_i) - \sigma(x_{i-1})$ are locally defined. A straightforward computation shows that the pullback of α satisfies $\alpha^* \circ \delta = d_{PV} \circ \alpha^*$. It hence yields a surjective chain map. It thus induces a surjective map from the continuous cocycle cohomology of the pattern groupoid to PV-cohomology. As we know that the two cohomologies are isomorphic, α^* has to be an isomorphism at least in the case that the cohomology is finitely generated.

2.3.3 Weak PV-cohomology

We now consider the situation in which the abelian group A carries a metric δ w.r.t. which it is complete. Then $\mathcal{C}(\Xi^n, A)$ is a complete module w.r.t. the metric $\tilde{\delta}(f, g) := \sup_{\xi \in \Xi^n} \delta(f(\xi), g(\xi))$.

Lemma 2.7. *The PV-differential extends to a continuous map $d_{PV} : \mathcal{C}(\Xi^n, A) \to \mathcal{C}(\Xi^{n+1}, A)$.*

Proof. First observe that $\mathrm{T}_{\tilde{\sigma},i}$ is a partial translation. $\theta_{\tilde{\sigma},i}$ is essentially its pull-back and since domain and range of $\mathrm{T}_{\tilde{\sigma},i}$ are clopen $\theta_{\tilde{\sigma},i}(f)$ is continuous for continuous f. Since $\tilde{\delta}$ is translation invariant the pull back of a partial translation is a partial isometry. Hence $\theta_{\tilde{\sigma},i}$ is continuous. Since d_{PV} is a finite sum of $\theta_{\tilde{\sigma},i}$ the result follows. □

Definition 2.8. *The weak PV-cohomology is the cohomology of the so-called weak PV-complex $(\mathcal{C}(\Xi^*, A), d_{PV})$. It is denoted by $H^*_{PV}(\Gamma_0, \mathcal{C}(\Xi, A))$.*

While for discrete groups, like $\mathbb{Z}$, the strong and the weak PV-complex coincide they differ vastly in the case that A is a continuous group, like $\mathbb{R}$. Although the inclusion $\mathcal{C}_{lc}(\Xi^n, A) \subset \mathcal{C}(\Xi^n, A)$ is a chain map, there is no evident relation between the two cohomologies in the continuous case.

2.4 PE de Rham cohomology

In this section, we recall the definition of the pattern equivariant de Rham cohomology of a tiling.

Definition 2.9. *Let $\mathcal{P}$ be a tiling of finite local complexity. A function f on $\mathbb{R}^d$ is pattern equivariant with range $R > 0$ if*

$$\forall x, y \in \mathbb{R}^d,\ B_R[\mathrm{T}^x(\mathcal{P})] = B_R[\mathrm{T}^y(\mathcal{P})] \Rightarrow f(x) = f(y).$$

It is called strongly pattern equivariant (s-PE) if it is pattern equivariant with some finite range R.[2]

We denote by $\mathcal{A}^n_b(\mathbb{R}^d)$ the space of bounded (smooth) n-forms over $\mathbb{R}^d$. Such forms can be seen as smooth functions from $\mathbb{R}^d$ into $\Lambda^n \mathbb{R}^{d*}$, the exterior algebra of the dual of $\mathbb{R}^d$ (equipped with some norm). We may therefore consider $\mathcal{A}^n_{s-\mathcal{P}}(\mathbb{R}^d)$ the space of s-PE n-forms over $\mathbb{R}^d$. Together with the standard exterior derivative $(\mathcal{A}^*_{s-\mathcal{P}}(\mathbb{R}^d), d)$ is a subcomplex of the usual de Rham complex of differential forms on $\mathbb{R}^d$.

The cohomology of the differential complex $(\mathcal{A}^*_{s-\mathcal{P}}(\mathbb{R}^d), d)$ is called the strongly pattern equivariant cohomology and denoted by $H^*_{s-\mathcal{P}}(\mathbb{R}^d)$. As ordinary de Rham cohomology it is a cohomology with real coefficients. $H^*_{s-\mathcal{P}}(\mathbb{R}^d)$ is isomorphic to the Čech cohomology of the tiling space with real coefficients [10].

Strongly pattern equivariant functions are in some sense algebraic, namely the size R of the patch around a point x which has to be inspected to obtain the value of the function is fixed and finite. This naturally calls for looking at functions which are obtained as limits of strongly pattern equivariant functions. These are called weakly pattern-equivariant. More precisely, the space of smooth weakly pattern equivariant n-forms $\mathcal{A}^*_{w-\mathcal{P}}(\mathbb{R}^d)$ is the closure of $\mathcal{A}^*_{s-\mathcal{P}}(\mathbb{R}^d)$ in $\mathcal{A}^*_b(\mathbb{R}^d)$ with respect to the Fréchet topology given by the family of semi norms s_k,

$$s_k(f) = \sup_{|\kappa| \leq k} ||D^\kappa f||_\infty$$

[2] This definition does not only apply to tilings but to any kind of pattern of $\mathbb{R}^d$, but it captures what we want only in the case of finite local complexity.

where $\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{N}^n$, $|\kappa| = \sum_{i=1}^n \kappa_i$, $D^\kappa f = \frac{\partial^{\kappa_1}}{\partial x_1^{\kappa_1}} \cdots \frac{\partial^{\kappa_n}}{\partial x_n^{\kappa_n}} f$ and $||.||_\infty$ is the norm of the uniform convergence. The weakly pattern equivariant cohomology $H^*_{w-\mathcal{P}}(\mathbb{R}^d, \mathbb{R})$ is the cohomology of the differential complex $(\mathcal{A}^*_{w-\mathcal{P}}(\mathbb{R}^d), d)$. $H^*_{w-\mathcal{P}}(\mathbb{R}^d, \mathbb{R})$ is isomorphic to the tangential cohomology of the tiling space [10]. We shall provide below an example which confirms our expectation that the weak pattern equivariant cohomology of aperiodic tilings is always infinitely generated.

In the last section, we will be interested in deformations of tilings. These deformations are parameterized[3] (in [11]) by a group called the pattern equivariant mixed group. It is defined by the quotient

$$Z^1_{s-\mathcal{P}}(\mathbb{R}^d)/B^1_{w-\mathcal{P}}(\mathbb{R}^d) \cap Z^1_{s-\mathcal{P}}(\mathbb{R}^d)$$

of closed strongly PE 1 forms modulo those 1 forms which are weakly exact, $Z^1_{s-\mathcal{P}}(\mathbb{R}^d) = \mathcal{A}^1_{s-\mathcal{P}}(\mathbb{R}^d) \cap \ker d$ and $B^1_{w-\mathcal{P}}(\mathbb{R}^d) = d(\mathcal{A}^0_{w-\mathcal{P}}(\mathbb{R}^d))$.

2.5 PE simplicial cohomology

Since $\mathcal{P}$ is a simplicial tiling it defines a simplicial complex in $\mathbb{R}^d$. The concept of pattern equivariance can equally well be defined for cochains (with values in any abelian group A) over this complex. In fact, a cochain is s-PE if there is some $k > 0$ such that its value on $\langle\sigma\rangle$ depends only on the k-neighborhood of the simplex σ of $\mathcal{P}$. We denote the degree n s-PE cochains with coefficients in A by $C^n_{s-\mathcal{P}}(\mathcal{P}, A)$. Clearly the coboundary operator maps s-PE cochains to s-PE cochains and so we may consider what is called the s-PE simplicial cohomology, which is the simplicial cohomology of the complex $(C^*_{s-\mathcal{P}}(\mathcal{P}, A), d_S)$. We denote it by $H^*_{s-\mathcal{P}}(\mathcal{P}, A)$.

It has been shown in [17] that s-PE simplicial cohomology is isomorphic to the Čech cohomology of the tiling space.

Just as one can define weak PE de Rham cohomology by using a completion w.r.t. a natural family of semi-norms, one can define weak PE simplicial cohomology by means of completion in the (metric) topology of uniform convergence. More precisely we define the w-PE cochains $C_{w-\mathcal{P}}(\mathcal{P}, A)$ (with coefficients in A) as the closure of $C_{s-\mathcal{P}}(\mathcal{P}, A)$ in the set of bounded cochains $C_b(\mathcal{P}, A)$ w.r.t. to the metric $\tilde{\delta}(c, c') = \sup_\sigma \delta(c(\langle\sigma\rangle), c'(\langle\sigma\rangle))$. Clearly the simplicial coboundary-boundary operator is continuous in the topology and defines the w-PE complex $(C^*_{w-\mathcal{P}}(\mathcal{P}, A), d_S)$ whose cohomology we call the w-PE simplicial cohomology and denote by $H^*_{w-\mathcal{P}}(\mathcal{P}, A)$.

[3]Strictly speaking the parameterization is in terms of this group with coefficients in $\mathbb{R}^d$.

3 PE simplicial cohomology and PV cohomology

We start with comparing the strong versions of PE simplicial cohomology and PV cohomology. This material is essentially known though not written in this form. At the end of the section we compare the weak versions. This is new.

3.1 PV cohomology as DL cohomology

We need to discuss in some detail the construction of the isomorphism between DL cohomology and PV cohomology. It essentially establishes that PV-cochains yield a model for the direct limit of the simplicial cochains of the APG complexes.

We denote in this section the pull back of $\rho_{k+1\,k} : \Gamma_{k+1} \to \Gamma_k$ by $\pi_{k\,k+1}$. Recall that the direct limit of the directed system $C^n(\Gamma_k, A) \xrightarrow{\pi_{k\,k+1}} C^n(\Gamma_{k+1}, A)$ is a module (denoted $\varinjlim C^n(\Gamma_k, A)$) together with module maps π_k such that the diagram

$$\begin{tikzcd}
C^n(\Gamma_k, A) \arrow[rr, "\pi_{k\,k+1}=\rho^*_{k+1\,k}"] \arrow[dr, "\pi_k"'] & & C^n(\Gamma_{k+1}, A) \arrow[dl, "\pi_{k+1}"] \\
& \varinjlim_l C^n(\Gamma_l, A) &
\end{tikzcd}$$

commutes. It has moreover the universal property that, whenever another module M with module maps $\pi'_k : C^n(\Gamma_k, A) \to M$ yields a commuting diagram

$$\begin{tikzcd}
C^n(\Gamma_k, A) \arrow[rr, "\pi_{k\,k+1}"] \arrow[dr, bend right, "\pi'_k"'] & & C^n(\Gamma_{k+1}, A) \arrow[dl, bend left, "\pi'_{k+1}"] \\
& M &
\end{tikzcd}$$

then the π'_i factor uniquely through the direct limit [12]. The latter means that there exists a unique homomorphism $h : \varinjlim C^n(\Gamma_k, A) \longrightarrow M$ satisfying $h \circ \pi_k = \pi'_k$ for all $k \in \mathbb{N}$.

Define the module maps $\eta_k : C^n(\Gamma_k, A) \longrightarrow \mathcal{C}_{lc}(\Xi^n_\Delta, A)$ by

$$\eta_k(a_\sigma) = a_{\Xi(\sigma)}.$$

These maps induce a new commuting diagram

$$\begin{tikzcd}
C^n(\Gamma_k, A) \arrow[rr, "\pi_{kj}"] \arrow[dr, bend right, "\eta_k"'] & & C^n(\Gamma_j, A) \arrow[dl, bend left, "\eta_j"] \\
& \mathcal{C}_{lc}(\Xi^n, A) &
\end{tikzcd}$$

By the universal property of the direct limit there exists a unique homomorphism

$$\eta : \varinjlim_{l} C^n(\Gamma_l, A) \longrightarrow \mathcal{C}_{lc}(\Xi^n, A)$$

such that

$$\eta \circ \pi_k = \eta_k \quad k \in \mathbb{N}.$$

Lemma 3.1. *The maps η_k are one-to-one.*

Proof. Let ξ in $\Xi(\sigma_0)$ for some $\sigma_0 \in S_k^n$. Hence ξ contains a patch at the origin which is translationally congruent to the face with its k-neighborhood encoded by σ_0. Thus

$$a_{\Xi(\sigma)}(\xi) = \begin{cases} a \text{ if } \sigma = \sigma_0, \\ 0 \text{ if } \sigma \neq \sigma_0. \end{cases}$$

Let $\gamma \in C^n(\Gamma_k, A)$. Then the above shows that $\eta_k(\gamma)(\xi) = \gamma(\langle \sigma_0 \rangle)$. Letting ξ vary we see that $\eta_k(\gamma) = 0$ implies $\gamma = 0$. $\square$

Lemma 3.2. *For all $\gamma \in \mathcal{C}(\Xi^n, A)$ there is $k \in \mathbb{N}$ such that $\gamma \in \mathrm{im}(\eta_k)$.*

Proof. By Lemma 2.5 every element of $\mathcal{C}(\Xi^n, A)$ is a finite sum of $a_{\Xi(\sigma)}$, $a \in A$ and $\sigma \in \Gamma_k$ for some k. Since $a_{\Xi(\sigma)} \in \mathrm{im}(\eta_k)$ and $\mathrm{im}(\eta_k) \subset \mathrm{im}(\eta_{k+1})$ the statement follows. $\square$

Lemma 3.3. *The η_k intertwine the differentials, i.e. $\eta_k \circ d_S = d_{PV} \circ \eta_k$.*

Proof. Recall the definition of the simplicial coboundary operator d_S on Γ_k: Given $\sigma \in S_k^n$ we have

$$d_S(a_\sigma) = \sum_{\tau \in S_k^{n+1}} \sum_{i=0}^{n+1} (-1)^i \delta_{\sigma\, \partial_i \tau} a_\tau.$$

Hence

$$\eta_k \circ d_S(a_\sigma)(\xi) = \sum_{i=0}^{n+1} (-1)^i \delta_{\sigma\, \partial_i \sigma_\xi^k}$$

where σ_ξ^k is the k-collared simplex of ξ on 0. On the other hand

$$\begin{aligned} d_{PV} \circ \eta_k(a_\sigma) &= \sum_{\tau \in \mathcal{S}_0^{n+1}} \sum_{i=0}^{n+1} \theta_{\tau,i}(a_{\Xi(\sigma)}) \\ &= \sum_{\tau \in \mathcal{S}_0^{n+1}} \sum_{i=0}^{n+1} \imath \circ \mathrm{T}_{\tau,i}^*(a_{\Xi(\sigma)} \big|_{\Xi(\partial_i \tau)}). \end{aligned}$$

Now

$$
\begin{aligned}
\imath \circ \mathrm{T}^*_{\tau,i}(a_{\Xi(\sigma)}\,|_{\Xi(\partial_i\tau)})(\xi) &= \begin{cases} a_{\Xi(\sigma)}(\mathrm{T}_{\tau,i}(\xi)) & \text{if } \xi \in \Xi(\tau) \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} 1 & \text{if } \sigma^k_{\xi - x_{\tau,i}} = \sigma \text{ and } \sigma^0_\xi = \tau \\ 0 & \text{else} \end{cases}
\end{aligned}
$$

and since $\sigma^k_{\xi - x_{\tau,i}} = \sigma$ and $\sigma^0_\xi = \tau$ can happen iff $\partial_i \sigma^k_\xi = \sigma$ the claim follows. □

Recall that the cohomology of $\varinjlim C(\Gamma_k, A)$ is the direct limit $\varinjlim H(\Gamma_k, A)$ together with module maps $H(\pi_k) : H(\Gamma_k, A) \to \varinjlim H(\Gamma_k, A)$ such that $H(\pi_{k+1}) = H(\pi_k) \circ H(\pi_{k\,k+1})$.

Theorem 3.4 ([18])**.** *η is a module isomorphism. It induces an isomorphism in cohomology.*

Proof. As the η_k intertwine the differentials they induce homomorphisms in cohomology $H(\eta_k) : H(\Gamma_k, A) \to H_{PV}(\Gamma_0, \mathcal{C}_{lc}(\Xi, A))$ such that $H(\eta_{k+1}) = H(\eta_k) \circ H(\pi_{k\,k+1})$. By the universal property we get a unique homomorphism which we denote $H(\eta)$ making

$$
\begin{tikzcd}
H(\Gamma_k, A) \arrow[rr, "H(\pi_{kj})"] \arrow[dr, "H(\pi_k)"] \arrow[ddr, bend right, "H(\eta_k)"'] & & H(\Gamma_j, A) \arrow[dl, "H(\pi_j)"'] \arrow[ddl, bend left, "H(\eta_j)"] \\
& \varinjlim_l H(\Gamma_l, A) \arrow[d, "H(\eta)"] & \\
& H_{PV}(\Gamma_0, \mathcal{C}_{lc}(\Xi, A)) &
\end{tikzcd}
$$

commute. Hence $H(\eta) \circ H(\pi_k)([\omega]) = H(\eta_k)([\omega]) = [\eta_k(\omega)] = [\eta \circ \pi_k(\omega)]$ where we have denote by $[\cdot]$ cohomology classes.

Injectivity: Let $x \in \varinjlim H(\Gamma_l, A)$ such that $H(\eta)(x) = 0$. There exists $k \in \mathbb{N}$ and a cochain in Γ_k such that $x = H(\pi_k)[\omega]$. Thus $0 = [\eta_k(\omega)]$. Let $\gamma \in \mathcal{C}_{lc}(\Xi, A)$ satisfy $\eta_k(\omega) = d_{PV}\gamma$. By Lemma 3.2, there is $l \in \mathbb{N}$ and $\alpha \in C(\Gamma_l, A)$, such that $\gamma = \eta_l(\alpha)$. By Lemma 3.3 $d_{PV}\eta_l(\alpha) = \eta_l(d_S\alpha)$. We may suppose that $l \geq k$. Then the above implies that $\eta_l(\omega) = \eta_l(d_S\alpha)$ and hence, by Lemma 3.1 $\omega = d_S\alpha$.

Surjectivity: Let $[\gamma] \in H(\Gamma_0, \mathcal{C}_{lc}(\Xi, A))$. Then $\gamma = \eta(\tilde{\omega})$ for some $\tilde{\omega} \in \varinjlim C(\Gamma_l, A)$. There exists $k \in \mathbb{N}$ and a cochain $\omega \in \Gamma_k$ such that $\tilde{\omega} = \pi_k[\omega]$. Hence $[\gamma] = [\eta \circ \pi_k(\omega)] = H(\eta) \circ H(\pi_k)([\omega])$. □

3.2 PE simplicial as DL cohomology

Recall the definition (2.2) of $\tilde{\rho}_k : \mathbb{R}^d \to \Gamma_k$. In particular, to each simplex σ of $\mathcal{P}$ corresponds a simplex $\tilde{\rho}_k \circ \sigma$ ofΓ_k. Pulling these back over cochains we obtain module maps $\check{\rho}_k^* : C^*(\Gamma_k, A) \to C^*_{s-\mathcal{P}}(\mathcal{P}, A)$

$$\check{\rho}_k^*(c)(\langle\sigma\rangle) = c(\langle\tilde{\rho}_k \circ \sigma\rangle).$$

They induce a new commuting diagram

$$\begin{tikzcd} C^n(\Gamma_k, A) \arrow[rr, "\pi_{kj}"] \arrow[dr, "\check{\rho}_k^*"'] & & C^n(\Gamma_j, A) \arrow[dl, "\check{\rho}_j^*"] \\ & C^n_{s-\mathcal{P}}(\mathcal{P}, A) & \end{tikzcd}$$

By the universal property of the direct limit there exists a unique homomorphism

$$\check{\rho}^* : \varinjlim_l C^n(\Gamma_l, A) \longrightarrow C^n_{s-\mathcal{P}}(\mathcal{P}, A)$$

such that for all $k \in \mathbb{N}$,

$$\check{\rho}^* \circ \pi_k = \rho_k^*.$$

As Sadun explains in [17], every s-PE cochain on $\mathcal{P}$ can be viewed as the pull back of a cochain on Γ_k provided k is sufficiently large and vice versa. In other words, the analogue of Lemma 3.1 and Lemma 3.2 are true and $\check{\rho}^*$ is an isomorphism. It is trivial that the ρ_k^* intertwine the differentials as they are defined in essentially the same way. Hence the same proof as that for Thm. 3.4 yields that $\check{\rho}^*$ induces an isomorphism between DL cohomology and s-PE simplicial cohomology.

3.3 PE simplicial and PV-cohomology

The last two sections can be summarized by saying that the strong versions of PE simplicial and PV-cohomology are abstractly the same thing: in both cases the cochains arise as models for the direct limit of the cochains of APG-complexes. In fact, by the universal property the module morphism $\phi : C_{s-\mathcal{P}}(\mathcal{P}, A) \to \mathcal{C}_{lc}(\Xi, A)$:

$$\phi = \eta \circ (\check{\rho}^*)^{-1}$$

is an isomorphism which, by Lemma 3.3 and its analogon intertwines the differentials. It hence induces an isomorphism between the strong versions of PE simplicial and PV cohomology. By construction $\phi(\check{\rho}_k^*(a_\sigma)) = \eta_k(a_{\tilde{\sigma}}) = a_{\Xi(\tilde{\sigma})}$ and hence $\phi(c)$ is on the dense orbit through $\mathcal{P}$ given by

$$\phi(c)(\mathcal{P} - p) = c(\langle\sigma_p\rangle) \tag{3.1}$$

where σ_p is the simplex of $\mathcal{P}$ whose puncture is on p.

Lemma 3.5. *ϕ is an isometry and thus extends to an isomorphism, which we also denote by ϕ, between $C_{w-\mathcal{P}}(\mathcal{P},A)$ and $\mathcal{C}(\Xi,A)$.*

Proof. That ϕ is an isometry is direct. The continuous extension of a bijective isometry is bijective as well. $\square$

Corollary 3.6. *ϕ induces a isomorphism between the weakly pattern equivariant cohomology $H_{w-P}(\mathcal{P},A)$ and the weak PV-cohomology $H_{PV}(\Gamma_0,\mathcal{C}(\Xi,A))$.*

Proof. By continuity ϕ intertwines the differentials. Hence ϕ is a bijective chain map. $\square$

4 PE de Rham versus PE simplicial cohomology

4.1 A PE de Rahm theorem

The de Rham theorem for smooth manifolds states that, given a smooth triangulation of the manifold and hence a simplicial decomposition, then the de Rham cohomology of the manifold and the simplicial cohomology of the complex are isomorphic. Moreover, the isomorphism has a very simple form, it is induced from a chain map which associates to a k-form the k-cochain whose evaluation on the generator associated with a k-simplex is given by integrating the form over the simplex. A very careful exposition of the proof is given in [19] for the case of finite simplicial complexes in euclidean spaces.

We shall adapt this proof to obtain a PE version of the de Rham theorem the role of the manifold being played by $\mathbb{R}^d$ with its simplicial decomposition defined by the tiling $\mathcal{P}$. This then will lead to an explicit isomorphism between the s-PE de Rham cohomology of $\mathbb{R}^d$ and the s-PE simplicial cohomology of $\mathcal{P}$ with values in $\mathbb{R}$. We furthermore show that the maps involved are continuous so that the isomorphism extends to an explicit isomorphism between the weak versions.

Recall that $\mathcal{P}$ defines a smooth triangulation on $\mathbb{R}^d$ but even a smooth triangulation. This allows us to integrate a k-form $\omega \in \mathcal{A}_b^k(\mathbb{R}^d)$ over a k-simplex σ. In fact we write $\int_\sigma \omega$ to mean $\pm$ the integral of ω over the set imσ where the sign is $+$ provided the orientation of σ is the same than that induced on $\mathrm{im}(\sigma)$ by the (chosen) orientation of $\mathbb{R}^d$, and $-$ otherwise. We equip $C_b(\mathcal{P},\mathbb{R})$ with the sup-norm: $\|c\| = \sup_\sigma |c(\langle\sigma\rangle)|$. This makes it a complete real vector space on which the simplicial differential acts continuously.

Definition 4.1. *Let $J_l : \mathcal{A}_b^l(\mathbb{R}^d) \to C_b^l(\mathcal{P},\mathbb{R})$ be given by*

$$J_l(\omega)(\langle\sigma\rangle) = \int_\sigma \omega$$

for any l-simplex σ.

Our aim is to prove the following theorem

Theorem 4.2. *J_l induces an isomorphism between $H^l_{s-\mathcal{P}}(\mathbb{R}^d)$ and $H^l_{s-\mathcal{P}}(\mathcal{P},\mathbb{R})$ and also between $H^l_{w-\mathcal{P}}(\mathbb{R}^d)$ and $H^l_{w-\mathcal{P}}(\mathcal{P},\mathbb{R})$.*

We should say that the first part could also be derived from the de Rham type theorems for transversally locally constant tangential forms [10] or for branched manifolds [17], but these proofs use the Čech-de Rham double complex and hence yield a priori isomorphisms with Čech cohomology. We provide here a proof avoiding Čech cohomology and double complexes at all. So it is more elementary but also longer. But it has the advantage of extending directly to the topological closures (weak versions).

Lemma 4.3. *J_l has the following properties:*

1. $J_{l+1}\circ d = d_S \circ J_l$.
2. *J_l is continuous.*
3. *If ω is s-PE then $J_l(\omega)$ is s-PE.*
4. *If ω is w-PE then $J_l(\omega)$ is w-PE.*

Proof. The first statement is Stokes theorem. Continuity follows from $\|J_l(\omega)\| \le \sup_\sigma |\int_\sigma \omega| \le \|\omega\|_\infty \sup_\sigma vol(\sigma)$. The third statement is evident and implies the last one by continuity of J_l. □

The first step in showing a de Rham theorem is the construction of right inverse maps for J_l. We proceed as in [19].

Lemma 4.4. *There are module maps $\alpha_l : C^l_b(\mathcal{P},\mathbb{R}) \to \mathcal{A}^l_b(\mathbb{R}^d)$ which satisfy the following properties:*

1. $J_l \circ \alpha_l = id$.
2. $\alpha_{l+1}\circ d_S = d\circ \alpha_l$.
3. *α_l is continuous.*
4. *If c is s-PE then $\alpha_l(c)$ is s-PE.*
5. *If c is w-PE then $\alpha_l(c)$ is w-PE.*

Proof. We follow the construction of α_l given in [19]. It is based on a partition of unity $\{g_v\}_{v\in\mathcal{P}^{(0)}}$ which is subordinate to the covering given by the stars of the vertices v: The star $St(v)$ of v is simply the interior of the 1-neighborhood of v, i.e. the union over all open simplices touching v. Let F_v be the compact set points of $St(v)$ which have at least distance $\frac{1}{n+1}$ from the boundary of $St(v)$ and G_v the closed set of points which have at most distance $\frac{1}{n+2}$ from the complement of $St(v)$. As is well known there exists a smooth positive function $\check{g}_v$ which vanishes on G_v and equals 1 on F_v. Clearly we can require that the choice of $\check{g}_v$ be made so that if v and v' have the same 1-neighborhood up to translation then $\check{g}_v$ and $\check{g}_{v'}$ coincide up to translation by $v - v'$. Now $g_v = \check{g}_v / \sum_{v' \in \mathcal{P}^{(0)}} \check{g}_{v'}$ and α_l is given as follows. Let σ be a l-simplex and v_i the ith vertex, i.e. the image under σ of $x_i \in \Delta^l$. Then

$$\alpha_l(1_\sigma) = \omega_\sigma := l! \sum_{i=0}^{l} (-1)^i g_{v_i} dg_{v_0} \wedge \cdots \widehat{dg_{v_i}} \cdots dg_{v_l}.$$

A proof of the first two properties can be found in [19]. The above formula shows that $\alpha_l(1_\sigma) = \mathrm{T}^{x*}\alpha_l(1_{\sigma'})$ (T^{x*} denotes the pull back of T^x) if $\sigma = \mathrm{T}^x(\sigma')$ and their 2-neighborhoods coincide. $\alpha_l(c)$ is therefore s-PE provided c is a s-PE cochain.

To show continuity let $c = \sum_{\sigma \in \mathcal{P}^{(l)}} r_\sigma 1_\sigma$, $r_\sigma \in \mathbb{R}$. Then

$$s_k(\alpha_l(c)) \leq \sum_{|\kappa| \leq k} \|D^\kappa \sum_{\sigma \in \mathcal{P}^{(l)}} r_\sigma \omega_\sigma\|_\infty.$$

Since the support of ω_σ is contained in $St(v)$ and the number of l-simplices in a star is uniformly bounded, let's say by N_l, we have $\sup_x |D^\kappa \sum_{\sigma \in \mathcal{P}^{(l)}} r_\sigma \omega_\sigma(x)| \leq N_l \sup_{\sigma \in \mathcal{P}^{(l)}} |r_\sigma| \sup_x |D^\kappa \omega_\sigma(x)|$ and hence

$$s_k(\alpha_l(c)) \leq N_l \|c\| \max_{\sigma \in \mathcal{P}^{(l)}} s_k(\omega_\sigma).$$

Hence α_l is continuous from which the last statement follows by continuous extension. □

Corollary 4.5. *J_l induces a surjective homomorphism between $H^l_{s-\mathcal{P}}(\mathbb{R}^d)$ and $H^l_{s-\mathcal{P}}(\mathcal{P}, \mathbb{R})$ and also between $H^l_{w-\mathcal{P}}(\mathbb{R}^d)$ and $H^l_{w-\mathcal{P}}(\mathcal{P}, \mathbb{R})$.*

The proof injectivity of these maps is essentially based on Poincaré's lemma. In contrast to the usual case we need however good control on the contracting homotopies involved. Let U be an open star-shaped subset of $\mathbb{R}^d$ and $u \in U$ a center, i.e. for all $x \in U$ we have $\forall 0 \leq t \leq 1 : tx + (1-t)u \in U$. Then we denote by $h^{U,u}_{k-1} : \mathcal{A}^k_b(U) \to \mathcal{A}^{k-1}_b(U)$, $k > 0$, the following module map:

$$h^{U,u}_{k-1}(f dx_{i_1} \wedge \cdots dx_{i_k})(x) = I^u_{k-1}(f)(x) \sum_{j=1}^{k} (-1)^j (x_{i_j} - u_{i_j}) dx_{i_1} \wedge \cdots \widehat{dx_{i_j}} \cdots dx_{i_k}$$

where

$$I^u_{k-1}(f)(x) = \int_0^1 t^{k-1} f(tx + (1-t)u) dt.$$

Lemma 4.6. *Let U be bounded and as above. $h_k^{U,u}$ have the following properties*

1. $h_k^{U,u} \circ d = d \circ h_{k-1}^{U,u}$ *for* $k \geq 1$,
2. $h_k^{U-x,u-x} = \mathrm{T}^{x*} \circ h_k^{U,u}$,
3. $h_k^{U,u}$ *is continuous.*

Proof. $h_{k-1}^{U,u}$ is a standard homotopy for U and the first statement (Poincaré lemma for U) proven for instance in [19]. The second statement is a direct consequence of the definition. As for the third, continuity of $h_{k-1}^{U,u}$ follows from continuity of I_{k-1}^u since the form $\omega(x) = \sum_{j=1}^k (-1)^j (x_{i_j} - u_{i_j}) dx_{i_1} \wedge \cdots \widehat{dx_{i_j}} \cdots dx_{i_k}$ does not depend on f and has bounded seminorms given that U is bounded. Now

$$\|D^\kappa I_{k-1}^u(f)\|_\infty = \left\| \int_0^1 t^{k-1+|\kappa|} (D^\kappa f)(tx + (1-t)u) dt \right\|_\infty \leq \|D^\kappa f\|_\infty$$

shows that I_{k-1}^u is indeed continuous. □

Let σ be an k-simplex. We denote by $[\sigma]_\epsilon$ the ϵ-neighborhood of $\mathrm{im}(\sigma)$ and by $[\partial\sigma]_\epsilon$ the ϵ-neighborhood of its boundary $\partial\mathrm{im}(\sigma)$. We denote by $\mathcal{Z}_b^r(U)$ the closed bounded r-forms on U. For $r \neq k$, $r, k \geq 1$ let

$$\mathcal{D}_\epsilon^r(\sigma) = \{(\omega, \tau) \in \mathcal{Z}_b^r([\sigma]_\epsilon) \oplus \mathcal{A}_b^{r-1}([\partial\sigma]_\epsilon) : \omega \big|_{[\partial\sigma]_\epsilon} = d\tau\}$$

and for $r = k \geq 1$ let

$$\mathcal{D}_\epsilon^k(\sigma) = \{(\omega, \tau) \in \mathcal{Z}_b^k([\sigma]_\epsilon) \oplus \mathcal{A}_b^{k-1}([\partial\sigma]_\epsilon) : \omega \big|_{[\partial\sigma]_\epsilon} = d\tau, \int_\sigma \omega = \int_{\partial\sigma} \tau\}.$$

Lemma 4.7. *There exists an $\epsilon > 0$ and a module map $\beta^r(\sigma) : \mathcal{D}_\epsilon^r(\sigma) \to \mathcal{A}_b^{r-1}([\sigma]_\epsilon)$ satisfying*

1. $\tau' = \beta^r(\sigma)(\omega, \tau)$ *coincides with τ on $[\partial\sigma]_\epsilon$,*
2. $d\tau' = \omega$,
3. *If $\mathrm{T}^x(\sigma)$ is a simplex then $\beta^r(\mathrm{T}^x(\sigma)) = \mathrm{T}^{x*} \circ \beta^r(\sigma)$,*
4. $\beta^r(\sigma)$ *is continuous.*

Proof. The proof is by induction. For that we need to consider a second module map. For $r \neq k$, $r \geq 0$, $k \geq 1$ let $\mathcal{E}_\epsilon^r(\sigma) = \mathcal{Z}_b^r([\partial\sigma]_\epsilon$ and for $r = k - 1 \geq 0$ let

$$\mathcal{E}_\epsilon^k(\sigma) = \{\omega \in \mathcal{Z}_b^k([\partial\sigma]_\epsilon : \int_{\partial\sigma} \omega = 0\}.$$

We aim to show that there exists a module map $\gamma^r(\sigma) : \mathcal{E}_\epsilon^r(\sigma) \to \mathcal{Z}_b^r([\sigma]_\epsilon)$ which satisfies

1. $\gamma^r(\sigma)(\omega)$ coincides with ω on $[\partial\sigma]_{\frac{\epsilon}{2}}$,

2. If $\mathrm{T}^x(\sigma)$ is a simplex then $\gamma^r(\mathrm{T}^x(\sigma)) = \mathrm{T}^{x*} \circ \gamma^r(\sigma)$,

3. $\gamma^r(\sigma)$ is continuous.

$\gamma^0(\sigma)$ Let $\omega \in \mathcal{E}^0_\epsilon(\sigma)$. Since $d\omega = 0$, ω is constant on any connected subset of $\partial\sigma$. If $k \neq 1$ $\partial\sigma$ is connected and hence ω constant. If $k = 1$ the extra condition $\int_\sigma \omega = 0$ implies that ω is constant. Thus for all choices of k we can define $\gamma^0(\sigma)(\omega)$ to be the constant extension of ω to σ. The properties stated are obviously satisfied.

$\beta^r(\sigma)$ We assume the existence of $\gamma^{r-1}(\sigma)$ satisfying the above properties. This is certainly correct for $r = 1$ and then will follow inductively in view of the next step. Let $(\omega, \tau) \in \mathcal{D}^r_\epsilon(\sigma)$. Let b_σ be the barycenter of σ and $h^\sigma = h^{[\sigma]_\epsilon, b_\sigma}$. Define

$$\beta^r(\sigma)(\omega, \tau) := h^\sigma(\omega) - \gamma^{r-1}(\sigma)(h^\sigma(\omega)\big|_{[\partial\sigma]_\epsilon} - \tau).$$

The first two properties are verified in [19]. The other two follow immediately from the properties of $\gamma^{r-1}(\sigma)$ and Lemma 4.6.

$\gamma^r(\sigma)$ We assume the existence of $\beta^r(\sigma)$ satisfying the properties stated in the lemma. This follows inductively in view of the preceding step. Let $\omega \in \mathcal{E}^r_\epsilon(\sigma)$. Let v be the first vertex of σ (w.r.t. the ordering of the vertices) and σ' the corresponding $k-1$-simplex of the boundary of σ, i.e. the face which does not contain v. Then $[\partial\sigma\backslash\sigma']_\epsilon$ is star-shaped with center v. We let $\mu'_\sigma = h^{[\partial\sigma\backslash\sigma']_\epsilon, v}(\omega\big|_{[\partial\sigma\backslash\sigma']_\epsilon]})$ and $\mu''_\sigma = \beta^r(\sigma)(\omega\big|_{[\sigma']_\epsilon}, \mu_0\big|_{[\partial\sigma']_\epsilon})$. μ'_σ and μ''_σ coincide on their common domain and hence define an $r-1$-form μ_σ on $[\partial\sigma]_\epsilon$. It follows from Lemmata 2.1 and 4.6 and the induction hypothesis for β^r that $\mu_{\mathrm{T}^x(\sigma)} = \mathrm{T}^{x*}\mu_\sigma$ provided $\mathrm{T}^x(\sigma)$ is a simplex of the tiling. Now consider a family of smooth functions $b_\sigma : [\sigma]_\epsilon \to \mathbb{R}$, defined for all k-simplices, which is s-PE in the sense that $b_{\mathrm{T}^x(\sigma)} = \mathrm{T}^{x*}b_\sigma$. We suppose that each b_σ is identically 1 on $[\partial\sigma]_{\frac{\epsilon}{2}}$ and identically 0 on the complement of $[\partial\sigma]_\epsilon$. Then

$$\gamma^r(\sigma)(\omega) := d(b_\sigma\mu_\sigma)$$

satisfies $\gamma^r(\sigma)(\omega) = \omega$ on $[\partial\sigma]_{\frac{\epsilon}{2}}$, as is shown in [19], and $\gamma^r(\mathrm{T}^x(\sigma)) = \mathrm{T}^{x*} \circ \gamma^r(\sigma)$. Continuity of $\gamma^r(\sigma)$ follows from the continuity of $h^{[\partial\sigma\backslash\sigma']_\epsilon, v}$, $\beta^r(\sigma)$ and d, and smoothness of b_σ.

□

Let $\mathcal{D}^r_\epsilon(\mathcal{P}^{(k)})$ be defined as $\mathcal{D}^r_\epsilon(\sigma)$ except replacing the simplex by the k-skeleton. If $r = k$ we demand the integral equation $\int_\sigma \omega = \int_{\partial\sigma} \tau$ to hold for all k-simplices. We then define $\beta^r : \mathcal{D}^r_\epsilon(\mathcal{P}^{(k)}) \to \mathcal{A}^{r-1}_b([\mathcal{P}^k)]_\epsilon)$ by glueing the $\beta^r(\sigma)(\omega\,\big|_{[\sigma]_\epsilon}, \tau\,\big|_{[\partial s]_\epsilon})$ defined for the k-simplices σ of $\mathcal{P}^{(k)}$ together which is possible as they agree with (ω, τ) along the ϵ-neighborhood of $\mathcal{P}^{(k-1)}$.

Corollary 4.8. *1. If $(\omega, \tau) \in \mathcal{D}^r_\epsilon(\mathcal{P}^{(k)})$ is s-PE then $\beta^r(\omega, \tau)$ is s-PE.*

2. If $(\omega, \tau) \in \mathcal{D}^r_\epsilon(\mathcal{P}^{(k)})$ is w-PE then $\beta^r(\omega, \tau)$ is w-PE.

Lemma 4.9. *Let $\omega \in \mathcal{Z}^l_b(\mathbb{R}^d)$, $l \geq 1$, and $J_l(\omega) = d_S c$. Then there exists $\tau \in \mathcal{A}^{l-1}_b(\mathbb{R}^d)$ such that $\omega = d\tau$. Moreover,*

1. *If ω and c are s-PE then τ can be chosen s-PE.*
2. *If ω and c are w-PE then τ can be chosen w-PE.*

Proof. We follow again [19] constructing τ by a composition of module maps

$$\tau = \tilde{\beta}_n \circ \cdots \circ \tilde{\beta}_0(\omega).$$

$\tilde{\beta}_0(\omega)$ is an $l-1$-form on the ϵ-neighborhood of $\mathcal{P}^{(0)}$. We may therefore define it through its restrictions to the connected components of $[\mathcal{P}^{(0)}]_\epsilon$, namely the restriction to $[v]\epsilon$ is given by

$$h^{[v]\epsilon}(\omega) - \delta_{l1}(h^{[v]\epsilon}(\omega)(v) - c(v)).$$

If $k > 0$ then

$$\tilde{\beta}_k(\omega) = \beta^l(\omega, \tilde{\beta}_{k-1}(\omega)) - \delta_{l-1\,k}\alpha_{l-1}(J_{l-1}(\beta^l(\omega, \tilde{\beta}_{k-1}(\omega)) - c).$$

This construction of τ follows exactly the lines of the proof of [19] and so the proof given there for the first statement applies also in our context. The two remaining statements are now obtained by finite iteration of the results of Lemmata 4.3, 4.4, 4.6 and Cor. 4.8. □

We thus have proved Theorem 4.2.

Corollary 4.10. *The map $\psi = \phi \circ J : \mathcal{A}_{w-\mathcal{P}}(\mathbb{R}^d) \to \mathcal{C}(\Xi, \mathbb{R})$ induces an isomorphism $H(\psi)$ between $H_{w-\mathcal{P}}(\mathbb{R}^d)$ and $H_{PV}(\Gamma_0, \mathcal{C}(\Xi, \mathbb{R}))$ which restricts to an isomorphism between $H_{s-\mathcal{P}}(\mathbb{R}^d)$ and $H_{PV}(\Gamma_0, \mathcal{C}_{lc}(\Xi, \mathbb{R}))$.*

Let $\omega \in \mathcal{A}^n_{w-\mathcal{P}}(\mathbb{R}^d)$. An explicit formula for $\psi(\omega) \in \mathcal{C}(\Xi^n, \mathbb{R})$ is given by

$$\psi(\omega)(\mathcal{P} - p) = \int_{\sigma_p} \omega,$$

where σ_p is the simplex of $\mathcal{P}$ whose puncture is on $p \in \mathcal{P}^{n,punc}$.

Proof. Combine Cor. 3.6 with Thm. 4.2 and Eq. 3.1. □

5 Some applications

5.1 Weak cohomology of cut & project tilings with dimension and codimension equal to one

We saw that the weak PE de Rham cohomology, the weak PE simplicial cohomology with real values and the weak PV cohomology with real values are all isomorphic. We refer to this cohomology simply as the weak tiling cohomology. The following calculation shows that this cohomology is infinitely generated even for the simplest aperiodic tilings of finite local complexity. Our calculation is based on an old result, namely that the transversally *continuous* tangential cohomology of a Kronecker foliation on a torus is infinitely generated in degree 1 [13]. We expect that weak tiling cohomology is always infinitely generated for aperiodic tilings of finite local complexity.

As explained in [18] for one dimensional tilings PV-cohomology is just the cohomology of the group $\mathbb{Z}$ with coefficients in $\mathcal{C}(\Xi^0, \mathbb{Z})$ where the action is induced by shifting the tiling by a tile (this is more precisely the first return map into the 0-dimensional Δ-transversal Ξ^0 of the $\mathbb{R}$-action by translation of the tilings). The same argument applies to weak PV-cohomology upon replacing $\mathcal{C}(\Xi^0, \mathbb{Z})$ by $\mathcal{C}(\Xi^0, \mathbb{R})$.

For cut & project tilings with dimension and codimension equal to one the dynamical system $(\Xi^0, \mathbb{Z})$ can be described as follows [4]: The cut & project tiling is determined by two data, θ and K. θ is an irrational number defining the slope of a line $E \subset \mathbb{R}^2$ w.r.t. a choice of orthonormal base. K is a subset of the orthocomplement of E which is supposed to be compact and the closure of its interior. These data define a one dimensional tiling of $E \cong \mathbb{R}$. Indeed the tiles are intervals whose end points are given by the ortho-projection onto E of all points in $\mathbb{Z}^2 \cap (K + E)$ ($\mathbb{Z}^2$ is the lattice generated by the base and it is supposed that no point of that intersection lies on the boundary of $K + E$). Since θ is irrational the tiling is aperiodic.

On the 0-dimensional Δ-transversal of this tiling Ξ^0 the group $\mathbb{Z}$ acts via the first return map of the $\mathbb{R}$-action by translation of the tilings. This dynamical system $(\Xi^0, \mathbb{Z})$ factors onto the dynamical system $(S^1 = \mathbb{R}/\mathbb{Z}, \mathbb{Z})$ given by the rotation around θ. The factor map $\mu : \Xi^0 \to S^1$ is almost one-to-one. The points where it is not one-to-one, the preimages of so-called cut points, form a union of orbits under the $\mathbb{Z}$-action. On these orbits μ is two-to-one and we denote such points by $a = (a^+, a^-)$.

Theorem 5.1. *The first weak tiling cohomology of a cut & project tiling with dimension and codimension equal to one is infinitely generated.*

Proof. By the last theorem and the above remark the first weak tiling cohomology is isomorphic to $H^1(\mathbb{Z}, \mathcal{C}(\Xi^0, \mathbb{R}))$, the cohomology of the group $\mathbb{Z}$ with values in $\mathcal{C}(\Xi^0, \mathbb{R})$. The pull back of the factor map $\mu : \Xi^0 \to S^1$ gives rise to a $\mathbb{Z}$-equivariant short exact sequence of $\mathbb{Z}$-modules

$$0 \to \mathcal{C}(S^1, \mathbb{R}) \xrightarrow{\mu^*} \mathcal{C}(\Xi^0, \mathbb{R}) \to Q \to 0$$

where $Q = \mathcal{C}(\Xi^0, \mathbb{R})/\mathcal{C}(S^1, \mathbb{R})$ naturally inherits a $\mathbb{Z}$-action. We claim that $H^0(\mathbb{Z}, Q) = 0$ so that the long exact sequence in group cohomology contains the short exact sequence

$$0 \to H^1(\mathbb{Z}, \mathcal{C}(S^1, \mathbb{R})) \to H^1(\mathbb{Z}, \mathcal{C}(\Xi^0, \mathbb{R})) \to H^1(\mathbb{Z}, Q) \to 0.$$

We prove the claim: For a pre-image of a cut point $a = (a^+, a^-) \in \Xi^0$ let $\tau_a : \mathcal{C}(\Xi^0, \mathbb{R}) \to \mathbb{R}$ be given by $\tau_a(f) = f(a^+) - f(a^-)$. This is a module homomorphism which vanishes on functions of $\mu^*(\mathcal{C}(S^1, \mathbb{R}))$ and hence extends to a homomorphism $\tau_a : Q \to \mathbb{R}$. $H^0(\mathbb{Z}, Q)$ is the group of invariant elements. Assuming that $H^0(\mathbb{Z}, Q)$ is non-trivial suppose that $f \in \mathcal{C}(\Xi^0, \mathbb{R})$ projects onto a non-trivial invariant element in Q. Then there exists an a such that $\tau_a(f) = \tau_a(f \circ \varphi) \neq 0$. Choose $0 < 4\epsilon < |\tau_a(f)|$. There exists $g \in \mathcal{C}_{lc}(\Xi^0, \mathbb{R})$ such that $\|f - g\|_\infty < \epsilon$. This implies that $|\tau_a(f \circ \varphi^n) - \tau_a(g \circ \varphi^n)| < 2\epsilon$, for all $n \in \mathbb{Z}$. Hence $|\tau_a(f) - \tau_a(g \circ \varphi^n)| < 2\epsilon < \frac{1}{2}|\tau_a(f)|$, for all $n \in \mathbb{Z}$, and therefore $|\tau_a(g \circ \varphi^n)| > \frac{1}{2}|\tau_a(f)|$, for all $n \in \mathbb{Z}$. It follows that g has infinitely many jumps. But by compactness of Ξ^0 the elements of $\mathcal{C}_{lc}(\Xi^0, \mathbb{R})$ have only finitely many jumps. This being a contradiction we must have $H^0(\mathbb{Z}, Q) = 0$.

Hence $H^1(\mathbb{Z}, \mathcal{C}(\Xi^0, \mathbb{R}))$ is an extension of $H^1(\mathbb{Z}, Q)$ by $H^1(\mathbb{Z}, \mathcal{C}(S^1, \mathbb{R}))$. Now by the results of Mostow [13][page 98f], $H^1(\mathbb{Z}, \mathcal{C}(S^1, \mathbb{R}))$ is infinitely generated, as θ is irrational. $\square$

5.2 The mixed group in PV-cohomology

There are two approaches to the deformation theory of tilings. The first is based on the description of tiling cohomology as the direct limit $\lim_{\to} H((\Gamma_l, \mathbb{R}^d)$ and asymptotic negligible cocycles [3], and the second on pattern equivariant 1-forms [11]. In the context of the second approach a deformation of a Delone set $\mathcal{P}$ into a Delone set $\mathcal{P}'$ is given by a bi-Lipschitz smooth function $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ which has a strongly pattern equivariant differential and satisfies $\mathcal{P}' = \varphi(\mathcal{P})$. Moreover, if two such functions differ on $\mathcal{P}$ by a constant vector (which amounts to a global translation) then they define the same deformation. The deformations of $\mathcal{P}$ are therefore parameterized by the elements of $Z^1_{s-\mathcal{P}}(\mathbb{R}^d, \mathbb{R}^d)/N^1_{s-\mathcal{P}}(\mathbb{R}^d, \mathbb{R}^d)$ where

$$N^1_{s-\mathcal{P}}(\mathbb{R}^d, \mathbb{R}^d) = \{d\varphi \in B^1_{s-\mathcal{P}}(\mathbb{R}^d, \mathbb{R}^d) | \varphi \text{ is constant on } \mathcal{P}\}.$$

Deformations of tilings can be seen as deformations of their set of punctures.

Denote by $\mathcal{B}_{||\cdot||_\infty}(\omega, \varepsilon)$ the open ε-ball around an element $\omega \in Z^1_{s-\mathcal{P}}(\mathbb{R}^d, \mathbb{R}^d)$ w.r.t. the uniform norm. For sufficiently small ϵ the elements of $Z^1_{s-\mathcal{P}}(\mathbb{R}^d, \mathbb{R}^d) \cap \mathcal{B}_{||\cdot||_\infty}(d\mathrm{id}, \varepsilon)$ define invertible deformations (in sense of [11]) of $\mathcal{P}$. If two such elements differ by an element of $B^1_{s-\mathcal{P}}(\mathbb{R}^d, \mathbb{R}^d)$ then their images on $\mathcal{P}$ are mutually locally derivable. If their difference is in $B^1_{w-\mathcal{P}}(\mathbb{R}^d, \mathbb{R}^d)$ then their images on $\mathcal{P}$ define pointed topological conjugate dynamical systems. The elements near the class of $d$id in the mixed quotient group

$$Z^1_{s-\mathcal{P}}(\mathbb{R}^d, \mathbb{R}^d)/B^1_{w-\mathcal{P}}(\mathbb{R}^d, \mathbb{R}^d) \cap Z^1_{s-\mathcal{P}}(\mathbb{R}^d, \mathbb{R}^d)$$

parameterize therefore the set of small deformations modulo bounded deformations which are in the same pointed conjugacy class.

The following corollary yields a description of the parameter space of deformations modulo topological conjugacy in terms of a mixed PV-cohomology group. We denote by $Z^*_{PV}(\Gamma_0, -)$ and $B^*_{PV}(\Gamma_0, -)$ PV coycles and PV coboundaries, respectively.

Corollary 5.2. *The pattern equivariant mixed quotient group* $Z^*_{s-\mathcal{P}}(\mathbb{R}^d, \mathbb{R}^d)/B^*_{w-\mathcal{P}}(\mathbb{R}^d, \mathbb{R}^d) \cap Z^*_{s-\mathcal{P}}(\mathbb{R}^d, \mathbb{R}^d)$ *is isomorphic to*

$$Z^*_{PV}(\Gamma_0, \mathcal{C}_{lc}(\Xi, \mathbb{R}^d))/B^*_{PV}(\Gamma_0, \mathcal{C}(\Xi, \mathbb{R}^d)) \cap Z^*_{PV}(\Gamma_0, \mathcal{C}_{lc}(\Xi, \mathbb{R}^d)).$$

Proof. $H^*(\psi) : H^*_{w-\mathcal{P}}(\mathbb{R}^d) \to H^*_{PV}(\Gamma_o, \mathcal{C}(\Xi, \mathbb{R}))$ is injective and hence its restriction to $Z^*_{s-\mathcal{P}}(\mathbb{R}^d)/B^*_{w-\mathcal{P}}(\mathbb{R}^d) \cap Z^*_{s-\mathcal{P}}(\mathbb{R}^d) \subset H^*_{w-\mathcal{P}}(\mathbb{R}^d)$ is injective as well. $H^*(\psi) : H^*_{s-\mathcal{P}}(\mathbb{R}^d) \to H^*_{PV}(\Gamma_o, \mathcal{C}_{lc}(\Xi, \mathbb{R}))$ is surjective and $\psi(B^*_{w,\mathcal{P}}(\mathbb{R}^d)) \subset B^*_{PV}(\Gamma_0, \mathcal{C}(\Xi, \mathbb{R}))$. It follows that the restriction of $H^*(\psi)$ to $Z^*_{s-\mathcal{P}}(\mathbb{R}^d)/B^*_{w-\mathcal{P}}(\mathbb{R}^d) \cap Z^*_{s-\mathcal{P}}(\mathbb{R}^d)$ is surjective.

□

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